

Problems with Expected Utility: Lecture XVI

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1 Allais' Paradox

2 Common Consequence Effect

Allais' Paradox

- Machina starts by describing how the expected utility formulation is linear in probabilities.
 - We start by assuming that there are three possible outcomes for a random variable $\{x_1, x_2, x_3\}$ such that $x_1 < x_2 < x_3$ with probabilities $P = (p_1, p_2, p_3)$.
 - Given this formulation

$$\sum_{i=1}^3 p_i = 1 \Rightarrow p_2 = 1 - p_1 - p_3 \quad (1)$$

- Which implies

$$\begin{aligned} U(x) &= p_1 U(x_1) + p_2 U(x_2) + p_3 U(x_3) \\ &= p_1 U(x_1) + (1 - p_1 - p_3) U(x_2) + p_3 U(x_3) \end{aligned} \quad (2)$$

- Continued

- Holding the payoffs in Equation 3.52 constant and differentiating with respect to the levels of probability yields

$$dU(x) = U(x_1) dp_1 + (-dp_1 - dp_3) U(x_2) + U(x_3) dp_3 = 0$$

$$[U(x_1) - U(x_2)] dp_1 = -[-U(x_2) + U(x_3)] dp_3$$

$$\frac{[U(x_2) - U(x_1)]}{[U(x_3) - U(x_2)]} = \frac{dp_3}{dp_1} \quad (3)$$

which given that $x_1 < x_2 < x_3$ and if $U'(x) > 0$ implies that any increase in p_3 relative to p_1 that holds the total probability constant (i.e., the probabilities still sum to one) yields more expected utility.

- Continued

- Completing the model, we construct a mean-preserving spread (i.e., we change the probability in such a way that leaves the mean of the risky alternative unchanged, but changes the variance).

$$\begin{aligned}\bar{x} &= \sum_{i=1}^3 x_i p_i = x_1 p_1 + x_2 p_2 + x_3 p_3 \\ &= x_1 p_1 + x_2 (1 - p_1 - p_2) + x_3 p_3\end{aligned}\tag{4}$$

$$d\bar{x} = x_1 dp_1 + x_2 (-dp_1 - dp_2) + x_3 dp_3 = 0$$

$$\frac{[x_2 - x_1]}{[x_3 - x_2]} = \frac{dp_3}{dp_2}$$

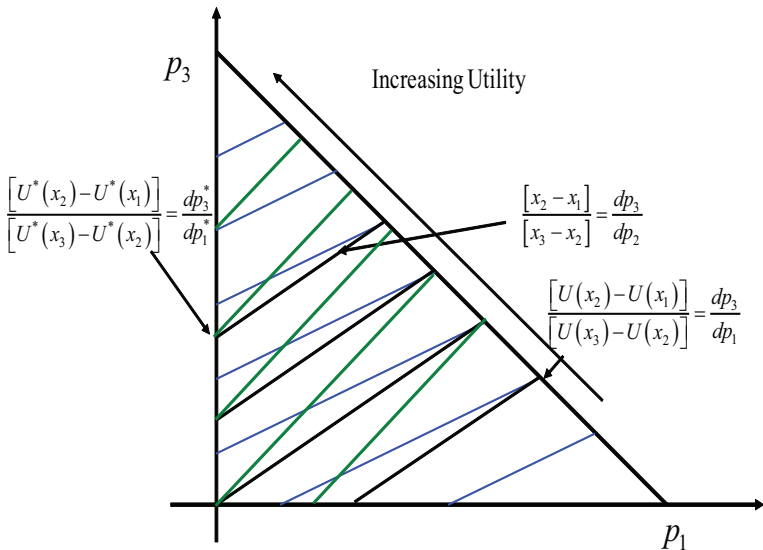


Figure: Linearity of Probabilities and Expected Utility

- The independence assumption places restrictions on individual's preferences over mixtures of lotteries.

$$P^* = (p_1^*, p_2^*, \dots, p_n^*) \quad P = (p_1, p_2, \dots, p_n)$$

$$\alpha P^* + (1 - \alpha) P = (\alpha p_1^* + (1 - \alpha) p_1, \dots, \alpha p_n^* + (1 - \alpha) p_n) \quad (5)$$

- Under linearity in probabilities

$$\begin{aligned} & \sum_{i=1}^n U(x_i) (\alpha p_i^* + (1 - \alpha) p_i) = \\ & \alpha \sum_{i=1}^n U(x_i) p_i^* + (1 - \alpha) \sum_{i=1}^n U(x_i) p_i \end{aligned} \quad (6)$$

- This structure allows for the development of the systematic violation known as Allais' Paradox. Consider the two pairs of gambles

$$\begin{aligned}
 a_1 &= \{ \text{1.00 chance of \$1,000,000} \\
 \text{versus } a_2 &= \left\{ \begin{array}{l} \text{0.10 chance of \$5,000,000} \\ \text{0.89 chance of \$1,000,000} \\ \text{0.01 chance of \$0} \end{array} \right. \\
 & \hspace{15em} (7) \\
 a_3 &= \left\{ \begin{array}{l} \text{0.10 chance of \$5,000,000} \\ \text{0.90 chance of \$0} \end{array} \right. \\
 \text{versus } a_4 &= \left\{ \begin{array}{l} \text{0.11 chance of \$1,000,000} \\ \text{0.89 chance of \$0} \end{array} \right.
 \end{aligned}$$

- Continued

- In this case, the selection of a_1 implies that a_4 is preferred to a_3 .
- However, numerous empirical studies have found a preference for a_1 and a_3 .
- Such preferences requires that the indifference curves "fan-out" so that the iso-utility curve becomes steeper as it approaches the p_3 axis and flatter as the iso-utility curve approaches the p_1 axis as depicted in the green lines in Figure 2.

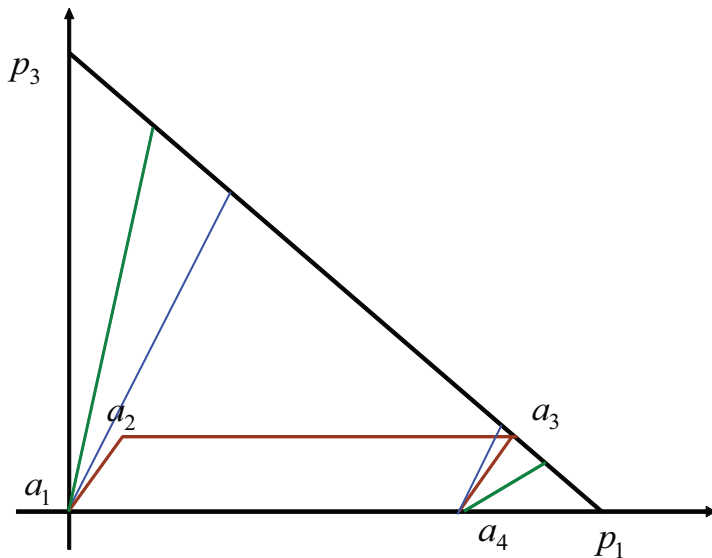


Figure: Fanning Out of Preferences

Common Consequence Effect

- The Allais Paradox is a special case of a general empirical pattern called the common consequence effect.
 - The common consequence effect involves pairs of probability mixtures of the form

$$\begin{aligned} b_1 : \alpha \delta_x + (1 - \alpha) P^{**} & \text{ versus } b_2 : \alpha P + (1 - \alpha) P^{**} \\ b_3 : \alpha \delta_x + (1 - \alpha) P^* & \text{ versus } b_4 : \alpha P + (1 - \alpha) P^* \end{aligned} \quad (8)$$

- Continued
 - Specific scenario

$$P = \begin{bmatrix} \frac{10}{11} & \frac{1}{11} \end{bmatrix}, Q = \begin{bmatrix} \$5,000,000 \\ \$0 \end{bmatrix}$$

$$\Rightarrow PQ = \$5,000,000 \times \frac{10}{11} + \$0 \times \frac{1}{11}$$

$$P^* = \begin{bmatrix} \frac{11}{11} & \frac{0}{11} \end{bmatrix}, Q^* = \begin{bmatrix} \$0 \\ K \end{bmatrix} \quad (9)$$

$$\Rightarrow P^*Q^* = \$0 \times \frac{11}{11} + K \times \frac{0}{11}$$

$$P^{**} = \begin{bmatrix} \frac{11}{11} & \frac{0}{11} \end{bmatrix}, Q^{**} = \begin{bmatrix} \$1,000,000 \\ K \end{bmatrix}$$

$$\Rightarrow P^{**}Q^{**} = \$1,000,000 \times \frac{11}{11} + K \times \frac{0}{11}$$

- Bet b_1

$$b_1 : \$1,000,000\alpha + (1 - \alpha) \times \frac{11}{11} \$1,000,000 = \$1,000,000 \quad (10)$$

- Bet b_2

$$\begin{aligned}
 b_2 : & \alpha \begin{bmatrix} \frac{10}{11} & \frac{1}{11} \end{bmatrix} \begin{bmatrix} \$5,000,000 \\ \$0 \end{bmatrix} + \\
 & (1 - \alpha) \begin{bmatrix} \frac{11}{11} & \frac{0}{11} \end{bmatrix} \begin{bmatrix} \$1,000,000 \\ K \end{bmatrix} \\
 = & \alpha \left(\frac{10}{11} \times \$5,000,000 + \frac{1}{11} \times \$0 \right) + \\
 & (1 - \alpha) \left(\frac{11}{11} \times \$1,000,000 + \frac{0}{11} \times K \right) \\
 = & \alpha \times \frac{10}{11} \$5,000,000 + (1 - \alpha) \$1,000,000
 \end{aligned} \tag{11}$$

- Bet b_3

$$\begin{aligned}
 b_3 &: \alpha \$1,000,000 + (1 - \alpha) \left[\begin{array}{cc} \frac{11}{11} & \frac{0}{11} \end{array} \right] \left[\begin{array}{c} \$0 \\ K \end{array} \right] \\
 &= \alpha \$1,000,000 + (1 - \alpha) \left(\frac{11}{11} \times \$0 + \frac{0}{11} K \right) \\
 &= \alpha \$1,000,000 + (1 - \alpha) \$0
 \end{aligned} \tag{12}$$

• Bet b_4

$$b_4 : \alpha \begin{bmatrix} \frac{10}{11} & \frac{1}{11} \end{bmatrix} \begin{bmatrix} \$5,000,000 \\ \$0 \end{bmatrix} + \\ (1 - \alpha) \begin{bmatrix} \frac{11}{11} & \frac{0}{11} \end{bmatrix} \begin{bmatrix} \$1,000,000 \\ K \end{bmatrix}$$

$$= \alpha \left(\frac{10}{11} \times \$5,000,000 + \frac{1}{11} \times \$0 \right) + \quad (13) \\ (1 - \alpha) \left(\frac{11}{11} \times \$1,000,000 + \frac{0}{11} \times K \right)$$

$$= \alpha \frac{10}{11} \times \$5,000,000 + \alpha \frac{1}{11} \times \$0 + (1 - \alpha) \$1,000,000$$