

Closed Form Solutions to Expected Utility: Lecture XIV

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September 9, 2026

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Closed Form Solutions

- By the von Neumann and Morgenstern proof we conclude that decision makers choose those decisions in a way that maximizes their expected utility.
 - We conclude that decision makers prefer more expected utility to less.
 - We use a variety of algebraic structures for utility that are positively monotonic and concave.
 - The Cobb-Douglas specification of utility yields an indirect utility function which is close to the power utility function.

$$U(Y) = \frac{Y^{1-r}}{1-r} \quad (1)$$

- To complete our problem we add information about the distribution function.
 - Following our discussion not expectations are closed form (i.e., have analytical solutions).
 - This is compounded by the form of the utility function.

- Continued

- Many of the closed form solutions we obtain are variants of the mean-variance specification or the expected utility of the gamble are functions of the mean and the variance of the distribution.
- One way to get a closed form solution is simply to choose a quadratic utility function.
 - The expectation of the quadratic utility function involves some function of the first two moments of the distribution (i.e., the mean and variance).

$$\begin{aligned}
 E[a + bY + cY^2] &= \int_{-\infty}^{\infty} (a + bY + cY^2) f(Y) dY \\
 &= a \int_{-\infty}^{\infty} f(Y) dY + b \int_{-\infty}^{\infty} Yf(Y) dY + c \int_{-\infty}^{\infty} Y^2 f(Y) dY \\
 &\quad a + B\mu_Y + c\mu_Y^2
 \end{aligned}$$

(2)

- Using a slightly different approach, the expected value of any utility function results in a mean-variance problem given that income is normally distributed.
 - The moment generating function for the normal distribution can be defined as

$$M_X(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right) \quad (3)$$

- Thus, since the normal distribution is completely defined by its first two moments, the expectation of any distribution function is a function of the mean and variance.

Negative Exponential Utility Function

- A more specific solution involves the use of the normal distribution function with the negative exponential utility function.
 - Starting with the negative exponential utility function

$$U(x) = -\exp(-\rho x) \quad (4)$$

- The expected utility can then be written as

$$\begin{aligned} E[U(x)] &= \int_{-\infty}^{\infty} -\exp[-\rho x] f(x; \mu, \sigma^2) dx \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} -\exp[-\rho x] \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx \end{aligned} \quad (5)$$

- Combining the exponential terms

$$E[U(x)] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left[-\frac{1}{2} \left(\frac{x - \mu}{\sigma} \right)^2 - \rho x \right] dx \quad (6)$$

- Continue

- Next transform the variables to the standard normal

$$z = \frac{x - \mu}{\sigma} \quad (7)$$

The inverse mapping for this transformation becomes

$$x = z\sigma + \mu \quad (8)$$

and the Jacobian is

$$dx = \sigma dz \quad (9)$$

The transformed expectation can then be expressed as

$$\begin{aligned} z = \frac{x - \mu}{\sigma} &\Rightarrow z\sigma = x - \mu \Rightarrow x = \mu + z\sigma \\ \therefore E[U(z)] &= -\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma \exp\left[-\frac{1}{2}z^2 - \rho z\sigma - \rho\mu\right] dz \end{aligned} \quad (10)$$

- Continued
 - Next, we complete the square

$$-\frac{1}{2}z^2 - \rho\sigma z - \rho\mu = \left\{ -\frac{1}{2}z^2 - \rho\sigma z - \frac{1}{2}\rho^2\sigma^2 \right\} + \frac{1}{2}\rho^2\sigma^2 - \rho\mu \quad (11)$$

Back to the density function

$$\begin{aligned} E[U(z)] &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left[-\rho \left(\mu - \frac{\rho}{2}\sigma^2 \right) - \frac{1}{2} (z - \rho\sigma)^2 \right] dz \\ &= -\exp \left(-\rho \left(\mu - \frac{\rho}{2}\sigma^2 \right) \right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left(-\frac{1}{2} (z - \rho\sigma)^2 \right) dz \\ &\quad - \exp \left(-\rho \left(\mu - \frac{\rho}{2}\sigma^2 \right) \right) \end{aligned} \quad (12)$$

Power Utility Function and the Log-Normal

- Taking the expectation of the power utility function where income is distributed log-normally yields

$$E[U(Y)] = \frac{1}{\sigma\sqrt{2\pi}} \int_0^{\infty} \frac{Y^{1-r}}{1-r} \left(\frac{1}{Y} \exp \left[-\frac{1}{2\sigma^2} (\ln(Y) - \mu)^2 \right] \right) dY \quad (13)$$

- Taking the inverse of the original transformation

$$Y^{1-r} \Rightarrow \exp((1-r)\ln(Y)) \Rightarrow X = \ln(Y) \quad (14)$$

• Therefore

$$\begin{aligned}
 E[U(X)] &= \frac{1}{1-r} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{-X^2 + 2X\mu - \mu^2 + 2(1-r)\sigma^2 X}{2\sigma^2}\right) dX \\
 &= \frac{1}{1-r} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{-X^2 + 2X(\mu + (1-r)\sigma^2) - \mu^2}{2\sigma^2}\right) dX \\
 &= \frac{1}{1-r} \frac{1}{\sigma\sqrt{2\pi}} \\
 &\quad \times \int_{-\infty}^{\infty} \exp\left(-\frac{(X - (\mu + (1-r)\sigma^2))^2}{2\sigma^2} + \mu(1-r) + (1-r)^2 \frac{\sigma^2}{2}\right) dX \\
 &= \frac{1}{1-r} \exp\left[(1-r)\left(\mu + (1-r)\frac{\sigma^2}{2}\right)\right]
 \end{aligned}$$

(15)