

Von Neumann-Morgenstern - Proof II: Lecture XIII

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1 Separating Classes

- Class I
- Class II
- Limitations

2 Mapping to Utility Space

Separating Classes

- There must exist α_0 with $0 < \alpha_0 < 1$ which separates the classes.
 - Thus, α_0 will be such that for $\alpha < \alpha_0$ the resulting bundle is in Class I,
 - And if $\alpha > \alpha_0$ then the resulting set is in Class II.
- First consider α_0 in Class I.
 - Specifically, we start by trying to generate a new point such that $\alpha > \alpha_0$, but $w \prec w_0$.
 - In this case $(1 - \alpha_0) u_0 + \alpha_0 v_0 \prec w_0$
 - Using 3:B:e $u \prec w \prec v \Rightarrow \alpha u + (1 - \alpha) v \prec w$ for some α

$$\gamma((1 - \alpha_0) u_0 + \alpha_0 v_0) + (1 - \gamma) v_0 \prec w_0 \quad (1)$$

since $w \prec v_0$.

- Continued

- Therefore by the combining axiom

$$\begin{aligned}
 & \gamma((1 - \alpha_0) u_0 + \alpha_0 v_0) + (1 - \gamma) v_0 \\
 & \Rightarrow \gamma u_0 - \gamma \alpha_0 u_0 + \gamma \alpha_0 v_0 + v_0 - \gamma v_0 \\
 & \Rightarrow \gamma(1 - \alpha_0) u_0 + (1 - \gamma(1 - \alpha_0)) v_0
 \end{aligned} \tag{2}$$

- Hence $\alpha = 1 - \gamma(1 - \alpha_0) \leq \alpha_0 \ni 0 < \gamma < 1$ belongs to I.
- This forms the contradiction, so that w cannot be preferred to w_0 if $\alpha < \alpha_0$.

- Second, consider α_0 in Class II.
 - Like the scenario above, we begin by generating the counter point that $\alpha < \alpha_0$, but $w \succ w_0$.
 - Then

$$(1 - \alpha_0) u_0 + \alpha_0 v_0 \succ w_0 \quad (3)$$

again substituting a linear combination of α_0 for the bundle w .

- Applying 3:B:d

$$u \succ w \succ v \Rightarrow \alpha u + (1 - \alpha) v \succ w \quad (4)$$

for some α with $0 < \alpha < 1$.

$$\gamma((1 - \alpha_0) u_0 + \alpha_0 v_0) + (1 - \gamma) u_0 \succ w_0 \quad (5)$$

- Continued

- Again, we combine applying the combining axiom

$$\begin{aligned}
 & \gamma((1 - \alpha_0) u_0 + \alpha_0 v_0) + (1 - \gamma) u_0 \\
 \Rightarrow & \cancel{\gamma u_0} - \gamma \alpha_0 u_0 + \gamma \alpha_0 v_0 + u_0 - \cancel{\gamma u_0} \\
 \Rightarrow & (1 - \gamma \alpha_0) u_0 + \gamma \alpha_0 v_0
 \end{aligned} \tag{6}$$

- Note that as $\gamma \rightarrow 1$ this expression becomes $w \approx w_0$.
- However, note that the convergence is from above.
- Specifically, if $\gamma < 1$ there is more weight on the $(1 - \gamma \alpha_0)$ term ($-\gamma \alpha_0$ becomes less negative).
- Letting $\alpha = \gamma \alpha_0$.

$$(1 - \alpha) u_0 + \alpha v_0 \succ w_0 \tag{7}$$

- Thus, $\alpha > \alpha_0$ which contradicts the original conjecture that $\alpha \leq \alpha_0$.

- There are some limitations.
 - First the numerical representation is for a utility interval $u_0 < w < v_0$, not for all w simultaneously.
 - Second, the numerical representation of **A:B** and **A:C** has not been correlated with our requirement **3:1:a** and **3:1:b**:
 - **3:1:a** $u \succ v \rightarrow V(u) > V(v)$
 - **3:1:b** $V(\alpha u + (1 - \alpha)v) = \alpha V(u) + (1 - \alpha)V(v)$

Mapping to Utility Space

- A. **A:D** Let u_0, v_0 such that $u_0 \prec v_0$. For all w in the interval $u_0 \prec w \prec v_0$ define the numerical function

$$f(w) = f_{u_0, v_0}(w) \quad (8)$$

f defined on the interval $[u_0, v_0]$ as

1. $f(u_0) = 0$
 2. $f(v_0) = 1$
 3. $f(w) = \alpha$ for $w \not\approx u_0, v_0$. Corresponding to the α that divides the set of bundles into Class I and II.
- B. **A:E** This mapping ($w \rightarrow f(w)$) has the following properties
1. It is monotone
 2. For $0 < \beta < 1$ and $w \not\approx u_0$ $f((1 - \beta)u_0 + \beta w) = \beta f(w)$
 3. For $0 < \beta < 1$ and $w \not\approx v_0$ $f((1 - \beta)u_0 + \beta w) = 1 - \beta + \beta f(w)$

- **A:D** is a definition, we must prove **A:E** and **A:F**.
- Starting with **A:E:1** $w \rightarrow f(w)$ is monotone.
- For $u_0 \prec w \prec v_0$ the mapping is monotone by **A:B**

$$\alpha \rightarrow w \approx (1 - \alpha) u_0 + \alpha v_0 \quad (9)$$

is a one-to-one and monotone mapping.

- Thus, all w of this interval are mapped onto numbers between 0 and 1 ($0 < \alpha < 1$).

- Thus, all w of this interval are mapped onto numbers between 0 and 1 ($0 < \alpha < 1$).
- **A:E:2** for $w \approx v_0$, $f((1 - \beta)u_0 + \beta v_0) = \beta$. For $w \not\approx v_0 \Rightarrow u_0 \prec w \prec v_0$, $w \approx (1 - \alpha_0)u_0 + \alpha v_0$.
- By **3:C:b**

$$\alpha(\beta u + (1 - \beta)v) + (1 - \alpha)v \approx \gamma v + (1 - \gamma)v \ni: \gamma = \alpha\beta \quad (10)$$

3:C:a $\alpha u + (1 - \alpha)v \approx (1 - \alpha)v + \alpha u.$

$$\begin{aligned} (1 - \beta)u_0 + \beta w &\approx (1 - \beta)u_0 + \beta((1 - \alpha)u_0 + \alpha v_0) \\ &\approx (1 - \beta\alpha)u_0 + \beta\alpha v_0 \end{aligned} \quad (11)$$

- This is the combining that we have used over and over again.

- Therefore by **A:B**

$$f((1 - \beta)u_0 + \beta v_0) = \alpha\beta = \beta\{\text{as } \alpha \rightarrow 1\} \quad (12)$$

- **A:1** Consider a mapping $w \rightarrow f_1(w)$ with **A:D:1**, **A:D:2**, and either **A:E:2** or **A:E:3**.
 - Defining the original mapping as $w \rightarrow f(w)$.
 - $\alpha \rightarrow \psi(\alpha)$ where $\psi(\alpha)$ is a point in preference space.
 - $\alpha \rightarrow f_1(\psi(\alpha)) = \phi(\alpha)$.
 - The first part of this expression $\alpha \rightarrow f_1(w)$ maps the probability measure from preference space into a function defined on the consumption bundle.
 - Next, we substitute in the inverse mapping set based on

$$w \rightarrow \alpha = f(w) \Leftarrow \alpha = \psi(\alpha) \quad (13)$$

- Since **A:1** and **A:2** fulfill **A:D:1** and **A:D:2** $\Rightarrow$ **A:5** $\phi(0) = 0$ and $\phi(1) = 1$.
- Since both **A:1** and **A:2** fulfill **A:E:2** and **A:E:3** $\Rightarrow$ **A:6**
 $\phi(\beta\alpha) = \beta\phi(\alpha)$ { by **A:E:2** }, and **A:7**
 $\phi(1 - \beta + \beta\alpha) = 1 - \beta + \beta\phi(\alpha)$ { by **A:E:3** }.

- Letting $\alpha \rightarrow 1$, $\{ \phi(\alpha\beta) = \beta\phi(\alpha), \phi(\alpha) \rightarrow 1 \}$
- **A:8** $\phi(\beta) = \beta$.

- **A:G** Let u_0, v_0 be fixed as above and let $\alpha_0 < \beta_0$ and w be fixed with $u_0 \preceq w \preceq v_0$. For all w in the interval define the numerical function $g(w) = g_{u_0, v_0}^{\alpha_0, \beta_0}(w)$

$$g(w) = (\beta_0 - \alpha_0) f(w) + \alpha_0 \quad (14)$$

- $g(u_0) = \alpha_0$.
- $g(v_0) = \beta_0$.