

Von Neumann-Morgenstern - Proof I: Lecture XII

Charles B. Moss

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1 Result A:A

2 Result A:B

3 Result A:C

Result A:A

- **A:A** If $u \prec v$ then $\alpha < \beta$ implies

$$(1 - \alpha) u + \alpha v \prec (1 - \beta) u + \beta v \quad (1)$$

- The direction of the assertion is that if $u \prec v$ and $\alpha < \beta$, then the preference ordering must follow.
- To demonstrate this we start with axiom 3:B:a given $0 \leq \alpha \leq 1$

$$u \prec v \Rightarrow u \prec \alpha u + (1 - \alpha) v \Rightarrow u \prec (1 - \beta) u + \beta v \quad (2)$$

- Continued

- Intuitively, this axiom states that if u is the inferior bundle, then any bundle constructed with any combination of v must be preferred to u .
- bf Axiom 3:B:b reverses this axiom by saying that if u is the preferred bundle then it must also be preferred to a bundle containing any amount of v .

$$u \succ v \Rightarrow u \succ \alpha u + (1 - \alpha) v \quad (3)$$

- We start from the first equation, replace with and by replacing the first in the right-hand side with preceding equation yields

$$(1 - \beta) u + \beta v \succ \gamma ((1 - \beta) u + \beta v) + (1 - \gamma) v \quad (4)$$

- Continued

- Next, we reverse the order of the combining in the first set of parenthesis in preceding yielding

$$(1 - \beta) u + \beta v \succ \gamma (\beta v + (1 - \beta) u) + (1 - \gamma) v \quad (5)$$

- Applying the combining axiom 3:C:b to the right hand side yields

$$\begin{aligned} (1 - \beta) u + \beta v &\succ \gamma u - \gamma \beta u + \gamma \beta v + u - \gamma u \\ &\succ (1 - \gamma \beta) u + \gamma \beta v \end{aligned} \quad (6)$$

Result A:B

- **A:B** Given two fixed points u_0 and v_0 with $u_0 \prec v_0$ consider the mapping

$$\alpha \rightarrow w \approx (1 - \alpha) u_0 + \alpha v_0 \quad (7)$$

This mapping is a one-to-one and monotone mapping of the interval $u_0 \prec w \prec v_0$.

- We said that α random variable X mapped from event space Ω into real number space

$$X : \Omega \rightarrow R \quad (8)$$

- In this case, α is a number that maps from preference into real number space such that the lottery is indifferent with a certain payment

$$\alpha : (1 - \alpha) u + \alpha v \approx w \rightarrow R \ni \alpha \in [0, 1] \quad (9)$$

- The mapping results follow the axioms

$$u \succ v \Rightarrow u \succ \alpha u + (1 - \alpha) v \quad (10)$$

such that $0 \leq \alpha \leq 1$ and

$$u \prec v \Rightarrow u \prec \alpha u + (1 - \alpha) v \quad (11)$$

- Given this mapping there exists a single number a such that for any number less than α

$$w \succ \alpha u_0 + (1 - \alpha) v_0 \quad (12)$$

and for any number greater than α

$$w \prec \alpha u_0 + (1 - \alpha) v_0 \quad (13)$$

Result A:C

- **A:C** The mapping **A:B** actually maps the α of $0 \leq \alpha \leq 1$ on all w of $u_0 \prec w \prec v_0$.
 - The proof is by contradiction.
 - If for some α in the region $0 \leq \alpha \leq 1$ where omitted then for all $0 \leq \alpha \leq 1$

$$\begin{aligned} (1 - \alpha) u_0 + \alpha v_0 &\not\succeq w_0 \\ (1 - \alpha) u_0 + \alpha v_0 &\succsim w_0 \end{aligned} \tag{14}$$

- We then divide the region into two classes

- Class I

$$(1 - \alpha) u_0 + \alpha v_0 \prec w_0 \quad (15)$$

- Class II

$$(1 - \alpha) u_0 + \alpha v_0 \succ w_0 \quad (16)$$

- Class I is that set of as that yield a bundle which is not preferred to w_0 . Class II is that set of as where the bundle constructed is preferred to w_0 .

- Class I is not empty: 3:B:c, $u \prec w \prec v$ implies the existence of

$$\alpha u + (1 - \alpha) v \prec w \quad (17)$$

- Continued

- Class I is not empty: $3:B:d$, $u \prec w \prec v$ implies the existence of

$$\alpha u + (1 - \alpha) v \succ w \quad (18)$$

- If α is in Class I and β is in Class II, then $\alpha < \beta$. The classes are disjoint $\alpha \neq \beta$. The alternative would be $\alpha > \beta$.
- What is the contradiction? If $\alpha \in I$ and then $\beta \in II$. The point of the proof is to demonstrate that $\alpha \not\prec \beta$.

- If

$$\alpha \Rightarrow (1 - \alpha) u_0 + \alpha v_0 \approx w_0 \quad (19)$$

and

$$\beta \Rightarrow (1 - \beta) u_0 + \beta v_0 \approx w_0^* \quad (20)$$

then $\alpha = \beta$ implies $w_0 \approx w_0^*$.

- Substituting $\alpha = \gamma\beta$ with $0 < \gamma < 1$ we have

$$w_0 \approx (1 - \beta\gamma) u_0 + \beta\gamma v_0 \quad (21)$$

- If $\gamma = 1$

$$w_0 \approx w_0^* \quad (22)$$

However, if $\gamma < 1$

$$w_0 \prec w_0^* \quad (23)$$

- Each counter proof then involves

$$(1 - \alpha_0) u_0 + \alpha_0 v_0 \prec w_0 \quad (24)$$

in Class I and

$$(1 - \alpha_0) u_0 + \alpha_0 v_0 \succcurlyeq w_0 \quad (25)$$

is in Class II.