

Expected Utility: Lecture X

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2 Expected Utility

Basic Utility

- A typical economic axiom is that economic agents (consumers, producers, etc.) behave in a way that maximizes their expected utility. The typical formulation is

$$\begin{aligned} \max_{x_1, x_2} U(x_1, x_2) \\ \text{s.t. } p_1 x_1 + p_2 x_2 \leq Y \end{aligned} \tag{1}$$

x_1 and x_2 are consumption goods and Y is monetary income. In decision making under risk, we are typically interested in the utility of income $U(Y)$. How do these concepts relate?

- The linkage between these two concepts is the indirect utility function which posits optimizing behavior by the economic agent. Specifically, assuming an Cobb-Douglas utility function the general utility maximization problem can be rewritten as

$$\begin{aligned} \max_{x_1, x_2} x_1^\alpha x_2^\beta \\ \text{s.t. } p_1 x_1 + p_2 x_2 \leq Y \end{aligned} \quad (2)$$

- Due to the concavity of the utility function, the inequality can be replaced with an equality. The maximization problem can then be reformulated as a Lagrangian

$$L = x_1^\alpha x_2^\beta + \lambda (Y - p_1 x_1 - p_2 x_2) \quad (3)$$

- The first order conditions are then

$$\begin{aligned}\frac{\partial L}{\partial x_1} &= \alpha \frac{x_1^{\alpha} x_2^{\beta}}{x_1} - \lambda p_1 = 0 \\ \frac{\partial L}{\partial x_2} &= \beta \frac{x_1^{\alpha} x_2^{\beta}}{x_2} - \lambda p_2 = 0 \\ \frac{\partial L}{\partial \lambda} &= Y - p_1 x_1 - p_2 x_2 = 0\end{aligned}\tag{4}$$

Taking the ratio of the first two first order conditions yields

$$x_2 = \frac{\beta}{\alpha} \frac{x_1 p_1}{p_2}\tag{5}$$

- Substituting this result into the third first order condition yields the demand for x_1 as a function of prices and income

$$x_1(p_1, p_2, Y) = \frac{Y}{p_1} \left(\frac{\alpha}{\alpha + \beta} \right) \quad (6)$$

Using this expression for x_1 in the result derived from the ratio of the first order conditions yields the Marshallian demand for x_2

$$x_2(p_1, p_2, Y) = \frac{Y}{p_2} \left(\frac{\beta}{\alpha + \beta} \right) \quad (7)$$

Indirect Utility Function

- The Marshallian demand curves can be used to derive the indirect utility function

$$V(p_1, p_2, Y) = \frac{Y^{\alpha+\beta}}{(\alpha + \beta)^{\alpha+\beta}} \left(\frac{\alpha}{p_1} \right)^{\alpha} \left(\frac{\beta}{p_2} \right)^{\beta} \quad (8)$$

This indirect utility function relates utility directly to income and prices assuming optimizing behavior.

Expenditure Function

- The expenditure function, like the indirect utility function, examines the implications of optimizing behavior. However, the expenditure function determines the minimum income required to provide a given level of utility. Mathematically, the expenditure function is given by the solution of

$$\begin{aligned} \min_{x_1, x_2} \quad & p_1 x_1 + p_2 x_2 \\ \text{s.t.} \quad & x_1^\alpha x_2^\beta \geq U^* \end{aligned} \tag{9}$$

The Lagrangian for this problem becomes

$$L = p_1 x_1 + p_2 x_2 + \mu \left(U^* - x_1^\alpha x_2^\beta \right) \tag{10}$$

- The first order conditions for this problem are then

$$\begin{aligned}\frac{\partial L}{\partial x_1} &= p_1 - \mu\alpha \frac{x_1^\alpha x_2^\beta}{x_1} = 0 \\ \frac{\partial L}{\partial x_2} &= p_2 - \mu\beta \frac{x_1^\alpha x_2^\beta}{x_2} = 0 \\ \frac{\partial L}{\partial \mu} &= U^* - x_1^\alpha x_2^\beta = 0\end{aligned}\tag{11}$$

Again, taking the ratio of the first two first order conditions yields

$$x_2 = \frac{p_1\beta}{p_2\alpha}\tag{12}$$

- Substituting this result into the third first-order condition yields

$$\begin{aligned}U^* - x_1^\alpha \left(\frac{p_1 \beta}{p_2 \alpha} x_1 \right)^\beta &= 0 \\U^* - x_1^{\alpha+\beta} \left(\frac{p_1 \beta}{p_2 \alpha} \right)^\beta &= 0 \\x_1^{\alpha+\beta} &= U^* \left(\frac{p_2 \alpha}{p_1 \beta} \right)^\beta\end{aligned}\tag{13}$$

The optimum level of x_1 given a fixed level of utility and prices (the Hicksian demand curve) is then

$$x_1^h(p_1, p_2, U) = U^{\frac{1}{\alpha+\beta}} \left(\frac{p_2 \alpha}{p_1 \beta} \right)^{\frac{\beta}{\alpha+\beta}}\tag{14}$$

- Similarly, the Hicksian demand for x_2 is

$$x_2^h(p_1, p_2, U) = U^{\frac{1}{\alpha+\beta}} \left(\frac{p_1 \beta}{p_2 \alpha} \right)^{\frac{\alpha}{\alpha+\beta}} \quad (15)$$

Substituting these expressions into the budget constraint yields the expenditure function

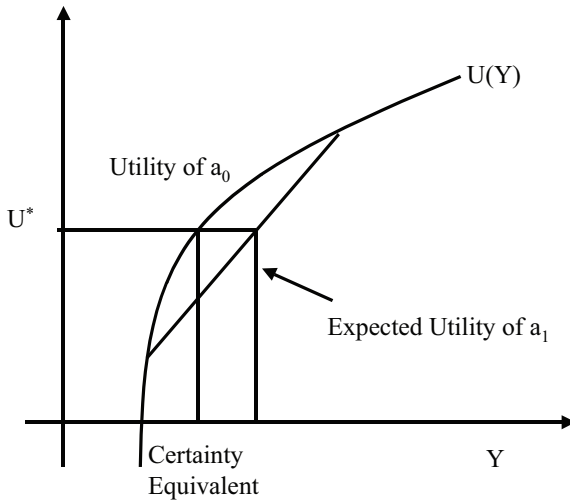
$$\begin{aligned} e(p_1, p_2, U) &= p_1 U^{\frac{1}{\alpha+\beta}} p_1^{-\frac{\beta}{\alpha+\beta}} p_2^{\frac{\beta}{\alpha+\beta}} \alpha^{\frac{\beta}{\alpha+\beta}} \beta^{\frac{\beta}{\alpha+\beta}} + \\ &\quad p_2 U^{\frac{1}{\alpha+\beta}} p_1^{\frac{\alpha}{\alpha+\beta}} p_2^{-\frac{\alpha}{\alpha+\beta}} \alpha^{-\frac{\alpha}{\alpha+\beta}} \beta^{\frac{\alpha}{\alpha+\beta}} \\ &= p_1^{\frac{\alpha}{\alpha+\beta}} U^{\frac{1}{\alpha+\beta}} p_2^{\frac{\beta}{\alpha+\beta}} \alpha^{\frac{\beta}{\alpha+\beta}} \beta^{-\frac{\beta}{\alpha+\beta}} + p_2^{\frac{\beta}{\alpha+\beta}} U^{\frac{1}{\alpha+\beta}} p_1^{\frac{\alpha}{\alpha+\beta}} \alpha^{-\frac{\alpha}{\alpha+\beta}} \beta^{\frac{\alpha}{\alpha+\beta}} \\ &= U^{\frac{1}{\alpha+\beta}} p_1^{\frac{\alpha}{\alpha+\beta}} p_2^{\frac{\beta}{\alpha+\beta}} \alpha^{-\frac{\alpha}{\alpha+\beta}} \beta^{-\frac{\beta}{\alpha+\beta}} (\alpha + \beta) \end{aligned} \quad (16)$$

- Note the duality between the functions. Starting with the expenditure function

$$\begin{aligned}e(p_1, p_2, U) &= U^{\frac{1}{\alpha+\beta}} p_1^{\frac{\alpha}{\alpha+\beta}} p_2^{\frac{\beta}{\alpha+\beta}} \alpha^{-\frac{\alpha}{\alpha+\beta}} \beta^{-\frac{\beta}{\alpha+\beta}} (\alpha + \beta) = Y \\U p_1^\alpha p_2^\beta \alpha^{-\alpha} \beta^{-\beta} (\alpha + \beta)^{\frac{1}{\alpha+\beta}} &= Y^{\frac{1}{\alpha+\beta}} \\U &= p_1^{-\alpha} p_2^{-\beta} \alpha^\alpha \beta^\beta (\alpha + \beta)^{\frac{1}{\alpha+\beta}} Y^{\frac{1}{\alpha+\beta}} \\V(p_1, p_2, Y) &= \left(\frac{\alpha}{p_1}\right)^\alpha \left(\frac{\beta}{p_2}\right)^\beta \frac{Y^{\frac{1}{\alpha+\beta}}}{(\alpha + \beta)^{\frac{1}{\alpha+\beta}}}\end{aligned}\tag{17}$$

Expected Utility

- The importance of these relationships for risk analysis involves the relationship between expected utility and the certainty equivalence function.
 - Next time we will discuss the concepts involved in expected utility analysis. For now, assume that economic agents behave to maximize their expected utilities.
 - Implicitly, this says that an economic agent is indifferent between two risky alternatives that yield the same expected utility.
 - If we let one of the alternatives be a risk-free alternative then the economic agent is indifferent between two alternatives a_0 and a_1 if the utility from the certain action equals the expected utility from the uncertain action.



- Simplifying the indirect utility function by ignoring the price term yields the power utility function

$$U(Y) = \frac{Y^{1-r}}{1-r} \quad (18)$$

Assume that a risky investment has a Bernoulli distribution paying \$150,000 with probability .6 and \$50,000 with probability .4. What is this investment worth? First, assume $r = 0.50$ then the expected utility of the gamble is

$$E[U(Y)] = 0.6 \frac{(150,000)^{1-0.5}}{1-0.50} + 0.4 \frac{(50,000)^{1-0.5}}{1-0.5} = 643.64 \quad (19)$$

- To compute the certainty equivalent of the gamble

$$U = \frac{Y^{1-r}}{1-r}$$

$$(1-r) U = Y^{1-r} \quad (20)$$

$$[(1-r) U]^{\frac{1}{1-r}} = Y$$

$$[(0.5) 643.64]^{\frac{1}{0.5}} = Y = 103,569.20$$

Note that the expected value of the gamble is \$110,000. This implies a risk premium of $110,000 - 103,569.20 = \$6,430.78$.