

# Martingales and Random Walks: Lecture IX

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## 1 Martingales

## 2 Stochastic Process

# Martingales

- Suppose that we have a sequence of random variables  $X_t : t = 1, 2, \dots$  (or  $X_1, X_2, X_3 \dots$ ) defined on a measure space  $(C, B, P)$ .
  - As with most of our models of risk, martingales have their basis in gambling concepts.
  - Let the random variable be the total winnings for the gambler after  $n$  games of chance  $(X_i : i = 1, 2, \dots)$ .
  - The question is then: what will the return be on the next game  $X_{n+1}$ .
  - Specifically, if the game is fair  $E[X_{n+1}]$  (also, the expected winnings on all the previous games is also zero).
  - Thus, if the game is fair this result is independent of past winnings. Mathematically

$$E[X_{n+1} | X_1, \dots, X_n] = X_n \quad (1)$$

- **Definition 2.18** *p-53* The sequence  $\{ (X_n, B_n), n = 1, 2, \dots \}$  is a *Martingale* if each  $n$  satisfies
  - $C_n \subset C_{n+1}$  or the Borel sets are nested. Thus, the set containing the current draw are contained in the set containing future draws.
  - $X_n$  is measurable on  $C_n$  or the sequence of random variables are defined on the nested Borel sets.
  - $E[|X_n|] < \infty$  or each expression is bounded.
  - $E[X_{n+1}|C_n] = X_n$ .
- As an example, consider Schmitz's (1995) discussion of farmland values.

$$\begin{aligned} V_t &= \sum_{s=1}^{\infty} \frac{E_t[CF_{t+s}|\Omega_t]}{(1+r)^2} \\ &= \frac{E_t[CF_{t+1}|\Omega_t]}{(1+r)} + \sum_{s=2}^{\infty} \frac{E_t[CF_{t+s}|\Omega_t]}{(1+r)^s} \end{aligned} \quad (2)$$

- Abstracting away from the expectations operator momentarily

$$\begin{aligned} V_t &= \frac{CF_{t+1}}{(1+r)} + \frac{1}{(1+r)} \sum_{s=2}^{\infty} \frac{CF_{t+s}}{(1+r)^2} \\ &= \frac{CF_{t+1}}{(1+r)} + \frac{1}{(1+r)} V_{t+1} \end{aligned} \quad (3)$$

- Solving Equation 3 we have

$$V_t - V_{t+1} = \frac{CF_{t+1}}{(1+r)} - rV_t = \epsilon_t \quad (4)$$

- So that if

$$\frac{CF_t}{(1+r)} = rV_t \quad (5)$$

there is no change in farmland values.

- Continued
  - Next, we add the random nature of cash flows back into our discussion using the nested Borel sets

$$\begin{aligned}v_t &= E[V_t | C_t] \\&= E\left[\sum_{s=1}^{\infty} \frac{CF_{t+s}}{(1+r)^s} \middle| C_t\right] \\&= \sum_{s=1}^{\infty} E\left[\frac{CF_{t+s}}{(1+r)^s} \middle| C_t\right]\end{aligned}\tag{6}$$

- Next, we walk the valuation ahead one period

$$E[v_{t+1} | C_t] = E\left[E\left[\sum_{s=2}^{\infty} \frac{CF_{t+s}}{(1+r)^{s-1}} \middle| C_{t+1}\right] \middle| C_t\right]\tag{7}$$

- Continued

- The cash flow for period  $CF_{t+2}$  (in the value  $v_{t+1}$ ) is discounted by a discount factor of

$$(1+r) \Leftarrow (1+r)^{s-1} \Big|_{s=2} \Leftarrow (1+r)^{2-1} \quad (8)$$

- Since  $C_n \subset C_{n+1}$  and, hence, by Definition 2.18(4) *p*-53

$$\mathbb{E} \left[ \mathbb{E} \left[ \sum_{s=2}^{\infty} \frac{CF_{t+s}}{(1+r)^{s-1}} \middle| C_{t+1} \right] \middle| C_t \right] = \mathbb{E} \left[ \sum_{s=2}^{\infty} \frac{CF_{t+s}}{(1+r)^{s-1}} \middle| C_t \right] \quad (9)$$

## • Continued

- Next, we add and subtract
- $CF_t$
- yielding

$$\begin{aligned} E[v_t | C_t] &= E \left[ \left( CF_t - CF_t + \sum_{s=2}^{\infty} \frac{CF_{t+s}}{(1+r)^{s-1}} \right) | C_t \right] \\ &= E \left[ \left( CF_t + \sum_{s=2}^{\infty} \frac{CF_{t+s}}{(1+r)^{s-1}} \right) | C_t \right] \times \frac{(1+r)}{(1+r)} - E[CF_t | C_t] \\ &= E \left[ \left( \frac{CF_t}{(1+r)} + \sum_{s=2}^{\infty} \frac{CF_{t+s}}{(1+r)^s} \right) | C_t \right] (1+r) - E[CF_t | C_t] \\ &= E \left[ \sum_{s=1}^{\infty} \frac{CF_t}{(1+r)^s} | C_t \right] (1+r) - E[CF_t | C_t] \end{aligned} \tag{10}$$

- Continued
  - Given this result

$$E[v_{t+1}|C_t] = (1 + r) v_t - E[CF_{t+1}|C_t] \quad (11)$$

- The result is a stochastic version of farmland valuation which is a martingale if

$$E[v_{t+1}|C_t] - v_t = rv_t - E[CF_{t+1}|C_t] \quad (12)$$

# Stochastic Process

- Building on this formulation the set of random variables described above can be rewritten as

$$\{X_t, t \in T\} \Rightarrow X_{t_i}(\omega) : t_1, t_2, \dots, t_n \in T \quad (13)$$

or that a set of random variables building on the measure theory introduced earlier.

- Thus, the probability of a given sequence of random variables can be defined as

$$f(X_{t_1}(\omega), X_{t_2}(\omega), \dots, X_{t_n}(\omega)) = P[\omega : X_{t_1}(\omega), X_{t_2}(\omega), \dots, X_{t_n}(\omega) \in \Omega] \quad (14)$$

- The most useful random process from our prospective is the Wiener process or the Brownian motion process defined as  $\{z_t, t \in [0, \infty)\}$  defined on a probability space  $(\Omega, \mathcal{C}, P)$  with the properties
  - $z_0(\omega) = 0$
  - For any two points along the line

$$P[z_{t_i} - z_{t_{i-1}} \in H_i \text{ for } i \leq n] = \prod_{i \leq n} P[z_{t_i} - z_{t_{i-1}} \in H_i] \quad (15)$$

- These increments are normally distributed

$$P[z_t - z_s \in H] = \frac{1}{\sqrt{2\pi(t-s)}} \int_H \exp\left[-\frac{1}{2} \frac{x^2}{(t-s)}\right] ds \quad (16)$$

# Readings

- Moss. 2010. *Risk, Uncertainty and the Agricultural Firm* – Chapter 3.