

Moment Generating Function and Method of Moments: Lecture VI

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Moment Generating Function

- Associated with each distribution is a unique function called the moment generating function that can be used to derive the moments of that distribution.
 - Definition 2.17** *p-33* The moment generating function $M_X(t)$ for the random variable X with distribution function $f(x)$ is defined as

$$M_X(t) = E[\exp(tx)] = \int_{-\infty}^{\infty} \exp[tx] f(x) dt \quad (1)$$

- If this moment generating function exists, the moments of the distribution are then defined by

$$\mu_r(x) = E[x^r] = M_X^{(r)}(0) = \left. \frac{d^r M_X(t)}{dt^r} \right|_{t=0} \quad (2)$$

- As an example, consider the moment generating function for the univariate normal distribution

$$\begin{aligned}
 M_X(t) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp[tx] \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx \\
 &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[\frac{2tx\sigma^2}{2\sigma^2} - \frac{x^2 - 2x\mu + \mu^2}{2\sigma^2}\right] dx \quad (3) \\
 &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[\frac{-x^2 + 2tx\sigma^2 + 2x\mu - \mu^2}{2\sigma^2}\right] dx
 \end{aligned}$$

- Grouping the middle term in the quadratic in we get

$$-x^2 + 2(t\sigma^2 + \mu)x - \mu^2 \quad (4)$$

- To solve this expression, we ask what has to be added, subtracted or multiplied to make this expression a perfect square (or quadratic function).

$$(t\sigma^2 + \mu)^2 = t^2\sigma^4 + 2t\mu\sigma^2 + \mu^2 \quad (5)$$

- Thus, we add and subtract $t^2\sigma^4 + 2t\mu\sigma^2$

$$\begin{aligned}
 M_X(t) &= \frac{1}{\sigma\sqrt{2\pi}} \\
 &\times \int_{-\infty}^{\infty} \exp \left[\frac{-x^2 + 2(t\sigma^2 + \mu)x - \mu^2 + t^2\sigma^4 + 2t\mu\sigma^2 - t^2\sigma^4 - 2t\mu\sigma^2}{2\sigma^2} \right] dx \\
 &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left[\frac{-x^2 + 2(t\sigma^2 + \mu)x - (t\sigma^2 + \mu)^2 + t^2\sigma^4 + 2t\mu\sigma^2}{2\sigma^2} \right] dx \\
 &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left[-\frac{(x - (t\sigma^2 + \mu))^2}{2\sigma^2} + \frac{1}{2}t^2\sigma^2 + t\mu \right] dx \\
 &\quad \exp \left[\frac{1}{2}t^2\sigma^2 + t\mu \right] \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left[-\frac{(x - (t\sigma^2 + \mu))^2}{2\sigma^2} \right] dx \\
 &\hspace{15em} (6)
 \end{aligned}$$

- Since the last term in the final equation in Equation 2.78 p-32 is simply the integral of a normal distribution with a different mean, we have

$$M_X(t) = \exp \left[\frac{1}{2} t^2 \sigma^2 + t\mu \right] \quad (7)$$

- To demonstrate the usefulness of this representation, we derive the first three moments of the normal distribution

$$\begin{aligned}
 M_X^{(1)}(0) &= \left. \frac{d \exp [t^2 \sigma^2 + \mu^2]}{dt} \right|_{t=0} \\
 &= (\sigma^2 t + \mu) \exp [t^2 \sigma^2 + \mu^2] \Big|_{t=0} = \mu \\
 M_X^{(2)}(0) &= \left. \frac{d (\sigma^2 t + \mu) \exp [t^2 \sigma^2 + \mu^2]}{dt} \right|_{t=0} \\
 &= \left(\sigma^2 \exp [t^2 \sigma^2 + \mu^2] + (\sigma^2 t + \mu)^2 \exp [t^2 \sigma^2 + \mu^2] \right) \Big|_{t=0} \\
 &= \sigma^2 + \mu^2
 \end{aligned}
 \tag{8}$$

$$\begin{aligned}
 M_X^{(3)}(0) &= \left. \frac{d \left(\sigma^2 \exp [t^2 \sigma^2 + \mu^2] + (\sigma^2 t + \mu)^2 \exp [t^2 \sigma^2 + \mu^2] \right)}{dt} \right|_{t=0} \\
 &= \left(\sigma^2 (\sigma^2 t + \mu) \exp [t^2 \sigma^2 + \mu^2] + \right. \\
 &\quad \left. 2 (\sigma^2 t + \mu) \exp [t^2 \sigma^2 + \mu^2] + (\sigma^2 t + \mu) \exp [t^2 \sigma^2 + \mu^2] \right) \Big|_{t=0} \\
 &= \left(3\sigma^2 (\sigma^2 t + \mu) \exp [t^2 \sigma^2 + \mu^2] + (\sigma^2 t + \mu)^3 \exp [t^2 \sigma^2 + \mu^2] \right) \Big|_{t=0} \\
 &= 3\sigma^2 \mu + \mu^3
 \end{aligned}
 \tag{9}$$

Estimating Probability Functions

- Estimating subjective probability distributions.
- Objective estimation of probability distributions.
 - Method of moments
 - Maximum likelihood
- Centering the Distribution
 - Time Trends
 - Production Functions

Subjective Probabilities

- The simplest approach is to elicit the triangular distribution function.
 - This approach amounts to asking the producer to specify the minimum, most likely, and maximum value of the random variable.
 - This procedure is particularly adept at modeling continuous random variables such as output prices with which the decision maker is fairly familiar.
 - However, the elicitation process does raise several efficiency problems. In its simplest form the distribution function is unconditioned.

- A second approach is to directly elicit the relative probability.
 - For example, we could divide the price of corn up into \$0.25/bushel increments as depicted in Table 1 and ask the farmer to assign a number about the relative likelihood of each outcome.
 - Summing the numbers and dividing each entry by the sum yields an subjective estimate of the probability density function.
 - In this case the probability weighted mean is 2.4754 and the probability weighted variance is 0.1244.
 - Note the similarity between the two distributions - The expected value of the triangular distribution from Equation 2.65 was 2.4333 and the variance for the triangular distribution is 0.0960.

Table: Direct Elicitation of Subjective Probabilities

Corn Price	Relative Merit	Probability
1.75	2	0.0488
2.00	8	0.1951
2.25	10	0.2439
2.50	9	0.2195
2.75	7	0.1707
3.00	6	0.1463
3.25	1	0.0244
Sum	41	1.0487

Objective Estimation of Probability Functions

- The method of moments approach builds on our forgoing discussion of the moments of a distribution.
 - Specifically in the method of moments we solve for the parameters that equate the sample moments with the theoretical moments.
 - Using the data presented in Table 2.3 *p-40* we compute the first and second moments of the sample.
 - We assume a distribution and solve the system of equations equating the sample moments with the theoretical moments

$$\begin{aligned}
 \mu_1(x) &= M_X^{(1)}(0) = \mu = 2.4123 \\
 &\Rightarrow \hat{\mu}_{MOM} = 2.4123 \\
 \mu_2(x) &= M_X^{(2)}(0) = \sigma^2 + \mu^2 = 5.9249 \\
 &\Rightarrow \hat{\sigma}_{MOM}^2 = 6.0041 - \hat{\mu}_{MOM}^2 = 0.1060
 \end{aligned}
 \tag{10}$$

- Implicit in this formulation is the concept of a sample. Most statistical approaches assume that individuals are drawn from the sample in the consistent with the probability density function.

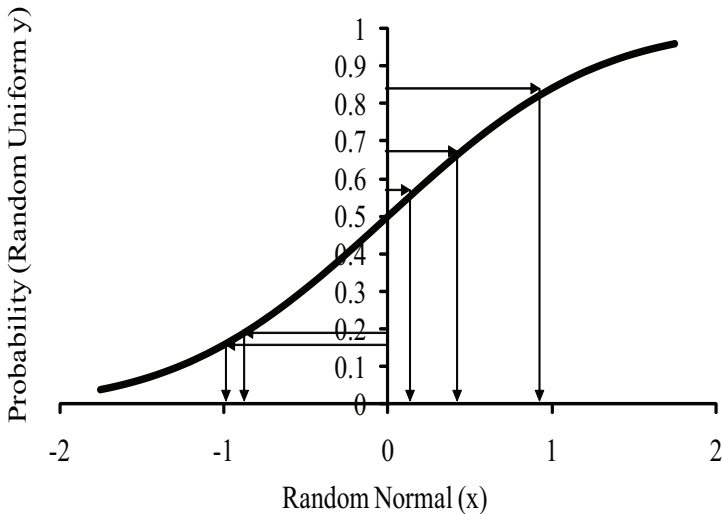
$$z = F(x^*) = \int_{-\infty}^{x^*} f(x) dx \Rightarrow z \in [0, 1] \quad (11)$$

$$x^* = F^{-1}(z)$$

- For our purposes, we envision drawing five random numbers from the uniform distribution and map them using the inverse mapping in Figure 1 into random normal space.
- For example, we draw 0.1631, 0.1882, 0.5868, 0.6580, 0.8136.

$$z = 0.1631 \Rightarrow x^* = -0.9820$$

$$z = 0.1182 \Rightarrow x^* = -0.8848 \quad (12)$$



- Continuing this mapping the vector 0.1631, 0.1882, 0.5868, 0.6580, 0.8136 implies the normal variables -0.9820, -0.8848, 0.2194, 0.4070, 0.8913.
- The normal sample is "closer to the middle" than the uniform sample - In the standard normal 66 percent of the probability mass lies within one standard deviation of the mean while in our transformed sample, 100% of the sample lies with one standard deviation of the mean (e.g., between -1 and 1).
- The crux of the argument is that the random process generating the outcome is assumed to generate the sample in such a way that more likely events are generated relatively more often. Thus, our equally weighted sample is assumed to replicate the original distribution.

- Given the data and mean moments in Table 2.3 *p*-40, we could fit the parameters of a gamma distribution

$$\mu_1(y) = \alpha\beta \Rightarrow \alpha = \frac{2.4123}{\beta}$$

$$\mu_2(y) = \alpha(1 + \alpha)\beta^2$$

$$\Rightarrow 2.4123 \left(1 + \frac{2.4123}{\beta}\right) \beta = 5.8192 + 2.4123\beta = 5.9249$$

$$\Rightarrow \hat{\beta}_{MOM} = 0.0438 \text{ and } \hat{\alpha}_{MOM} = 55.0493$$

(13)

- The data could explain two different distributions.
 - The question is then: which distribution is appropriate?
 - One way to solve this question is to examine the third moment of each specification.
 - The third empirical moment of the data presented in Table 2.3 is 14.7970.

$$\begin{aligned}
 \mu_3(x) &= \mu^3 + 3\mu\sigma^2 \Rightarrow 14.8047 \\
 \text{such that } \hat{\mu}_{MOM} &= 2.4123, \hat{\sigma}_{MOM}^2 = 0.1060 \\
 \mu_3(z) &= \alpha(1+\alpha)(2+\alpha)\beta^2 \Rightarrow 14.7909 \\
 \text{such that } \hat{\alpha}_{MOM} &= 55.0493, \hat{\beta}_{MOM} = 0.0438
 \end{aligned}
 \tag{14}$$

Readings

- Moss (2010) *Risk, Uncertainty and the Agricultural Firm* Sections 2.6.
- Moss (2014) *Mathematical Statistics for Applied Econometrics* Chapter 7.