

Expected Value and Moments: Lecture V

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Expected Value

- The random variable can also be described using a statistic.
 - One basic statistic encountered by students in statistics courses is the mean of a random variable.
 - Definition 2.12** *p-28* The expected value (expectation or mean) of a discrete random variable X , denoted $E[X]$, is defined as

$$E[X] = \sum_{x_i \in X} x_i P[x_i] \quad (1)$$

- Definition 2.13** *p-28* The expected value of a continuous random variable X is then defined as

$$E[X] = \int_{x \in X} xf(x) dx \quad (2)$$

- Taking the die roll as an example, if we let each side be equally likely the expected value of the roll of a fair die is

$$\begin{aligned} E[x] &= \sum_{i=1}^6 iP[i] = 1\frac{1}{6} + 2\frac{1}{6} + 3\frac{1}{6} + 4\frac{1}{6} + 5\frac{1}{6} + 6\frac{1}{6} \\ &= 3\frac{1}{2} \end{aligned} \quad (3)$$

- This result points out an interesting fact about the expected value, namely that the expected value need not be an element of the sample set.

- Suppose we weight the die so that it is no longer fair. Specifically, assume that $P[i] = 1/9$ for $i = \{1, 2, 5, 6\}$, $P[3] = 3/9$, and $P[4] = 2/9$, then the expected value becomes

$$\begin{aligned} E[X] &= \sum_{i=1}^6 iP[i] = 1\frac{1}{9} + 2\frac{1}{9} + 3\frac{3}{9} + 4\frac{2}{9} + 5\frac{1}{9} + 6\frac{1}{9} \\ &= \frac{31}{9} \end{aligned} \quad (4)$$

- Taking the uniform distribution as an example, the mean is derived as

$$\begin{aligned} E[X] &= \int_0^1 xf(x) dx = \int_0^1 x dx = \left(\frac{1}{2}x^2 \right) \Big|_0^1 \\ &= \frac{1}{2}(1 - 0) = \frac{1}{2} \end{aligned} \quad (5)$$

Empirical Moments

- A second definition of the mean is a sample based definition. In this formulation, each sample point is typically given an equal weight

$$\bar{X} = \frac{1}{N} \sum_{i=1}^N x_i \quad (6)$$

Table: Simulated Sample Die Rolls

Observation	Value
1	4
2	6
3	5
4	5
5	2
6	5
7	1
8	2
9	6
10	2

- The empirical or sample mean based in Table 1 is then computed as

$$\bar{X} = \frac{1}{10} \sum_{i=1}^{10} x_i = \frac{38}{10} = 3.8 \quad (7)$$

which is close to the theoretical sample mean presented in Equation 3.

Applications

- The mean has several statistical applications, but intuitively the mean represents the central tendency of the random variable.
 - The concept of the central tendency can be expanded past the value of the random variable itself.
 - **Definition 2.14** *p-29* The expected value of a function $g(x)$ can be expressed as

$$E[g(x)] = \int_{x \in X} g(x) f(x) dx \quad (8)$$

- In the study of the effect of risk on the agricultural firm we are frequently interested in the expected value of profit.
- Building on the triangular distribution function for the price of corn developed in the preceding section, we define the expected profit as

$$E [\pi (p_y)] = p_y y - c \quad (9)$$

- Expected profit can then be derived as

$$\begin{aligned}
 E[\pi(p_y)] &= \int_{1.75}^{3.25} [p_y y - c] f(p_y) dp_y \\
 &= y \int_{1.75}^{3.25} p_y f(p_y) dp_y - c \\
 &= y \left[\int_{1.75}^{2.30} p_y (a + bp_y) dp_y + \int_{2.30}^{3.25} p_y (c + dp_y) dp_y \right] - c \\
 &= y \left[\left(\frac{1}{2} ap_y^2 + \frac{1}{3} bp_y^3 \right) \Big|_{1.75}^{2.30} + \left(\frac{1}{2} cp_y^2 + \frac{1}{3} dp_y^3 \right) \Big|_{2.30}^{3.25} \right] - c \\
 &\quad y \times 2.433314 - c
 \end{aligned}
 \tag{10}$$

- This example is purely pedagogic in that profit is a linear function.

$$E[p_y] = \int_{1.75}^{3.25} p_y f(p_y) dp_y \quad (11)$$

$$E[\pi(p_y)] = yE[p_y] - c$$

$$\begin{aligned} E[p_y] &= \int_{1.75}^{2.30} p_y (a + bp_y) dp_y + \int_{2.30}^{3.25} p_y (c + dp_y) dp_y \\ &= \left(\frac{1}{2}ap_y^2 + \frac{1}{3}bp_y^3 \right) \Big|_{1.75}^{2.30} + \left(\frac{1}{2}cp_y^2 + \frac{1}{3}dp_y^3 \right) \Big|_{2.30}^{3.25} = 2.433314 \end{aligned} \quad (12)$$

Bivariate Random Variables

- **Definition 2.15** *p-31* The expected value of a bivariate function of random variables is defined as

$$E[h(x, y)] = \int_y \int_x h(x, y) f(x, y) dx dy \quad (13)$$

where $g(x, y)$ is a bivariate distribution function and the integral is bounded.

- In the forgoing example, if we let y be distributed uniform, under independence the joint distribution function ($g(p_y, y)$) becomes

$$g(p_y, y) = f(p_y) \times 1 \quad (14)$$

- Further because the random variables are independent

$$E[p_y y] = E[p_y] E[y] \quad (15)$$

- The expected value of profit is then

$$E[p_y y] = 2.433314 \times \frac{1}{2} = 1.216657 \quad (16)$$

Moments of a Distribution

- The r^{th} moment of a distribution is defined as

$$\mu_r(X) = E[X^r] = \int_{-\infty}^{\infty} x^r f(x) dx \quad (17)$$

- The first moment of a distribution is equal to the mean

$$\mu_1(X) = E[X^1] = \int_{-\infty}^{\infty} x^1 f(x) dx = E[X] \quad (18)$$

Moments of the Uniform Distribution

- Using the uniform distribution as an example, the first four moments of the distribution are

$$\begin{aligned}\mu_1(X) &= \int_0^1 x dx = \frac{1}{2} (x^2|_0^1 = \frac{1}{2} \\ \mu_2(X) &= \int_0^1 x^2 dx = \frac{1}{3} (x^3|_0^1 = \frac{1}{3} \\ \mu_3(X) &= \int_0^1 x^3 dx = \frac{1}{4} (x^4|_0^1 = \frac{1}{4} \\ \mu_4(X) &= \int_0^1 x^4 dx = \frac{1}{5} (x^5|_0^1 = \frac{1}{5}\end{aligned}\tag{19}$$

- The central moments of a distribution are defined as

$$\tilde{\mu}_r(X) = E[X - E[X]]^r = E[X - \mu_1(X)]^r \quad (20)$$

- The second central moment of a distribution defined as the variance of the distribution

$$V[X] = E[X - E[X]]^2 \quad (21)$$

Readings

- Moss (2010) *Risk, Uncertainty and the Agricultural Firm* Sections 2.6.
- Moss (2014) *Mathematical Statistics for Applied Econometrics* Chapter 7.