# Lecture XVII: Net Present Value and its Mechanics

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# Net Present Value - The Basic Mechanics

- We begin by defining the **net present value** of an investment opportunity as

$$NPV = -I + \sum_{i=1}^{N} \frac{NCF_i}{(1+r)^i} \tag{1}$$

where $NPV$ is the net present value of the investment opportunity (the wealth created by the investment), $I$ as the cost of the investment item, $NCF_i$ as the net cash flow created by the investment in period $i$, and $r$ as the weighted average cost of capital.

- To develop these generalizations, we develop the concept of a difference equation.
- A difference equation is basically an equation involving lagged values for one or more variables.

$$\begin{gathered} Y_t = aY_{t-1} + b \\ Y_0 = c \end{gathered} \tag{2}$$

  is a difference equation.
- The value of $Y_t$ in any period can be derived as

$$\begin{gathered} \left(Y_t - aY_{t-1}\right) = b \\ \left(1 - aL\right) Y_t = b \end{gathered} \tag{3}$$

  where $L$ is the lag operator.

- In most applications we assume that $-1 < a < 1$ so that the equation is "stationary".
- Solving the system by substitution

$$\begin{gathered} Y_1 = ac + b \\ Y_2 = (ac + b)\,a + b = a^2c + ab + b \\ Y_3 = \left(a^2c + ab + b\right)a + b = a^3c + a^2b + ab + b \\ Y_t = a^tc + \sum_{i=0}^{t-1} a^ib. \end{gathered} \tag{4}$$

- The present value is basically a difference equation.
- In the above example let $a = 1/\left(1+r\right)$. Hence, $Y_t = PV_t$ is

$$Y_t = \left(\frac{1}{1+r}\right)^t c + \sum_{i=0}^{t-1}\left(\frac{1}{1+r}\right)^i b \tag{5}$$

assuming $c$ is a terminal cash flow and $b$ is an average annual cash flow.

# Values of Streams of Constant Payments

- We will generalize the difference equation formulas presented above.
- Starting with a constant payment into the infinite future

$$\frac{1}{1-\lambda L} = \left(1 + \lambda L + \lambda^2 L^2 + \lambda^3 L^3 + \cdots\right) \text{ if } |\lambda| < 1. \quad (6)$$

  To develop this notion, start with the formulation $(K_0 + K_1 + K_2 + K_3 + \cdots)(1 - \lambda L)$ where we are trying to determine $K_0$, $K_1$, $K_2$, $\cdots$.
- We start with the guess that $K_0 = 1$ and focus on deriving the value of $K_1$ (i.e., holding $K_2$, $K_3$, $\cdots$ to zero).

$$\frac{1}{1-\lambda L} = (1 + K_1)(1 - \lambda L) \Rightarrow 1 - (1 - \lambda L) 1 = \lambda L. \quad (7)$$

- What is our best guess for the value of $K_1$? What about $K_1 = \lambda L$? Bringing $K_2$ back into the expression yields

$$\begin{array}{l} \frac{1}{1-\lambda L} = (1 + \lambda L + K_2)(1 - \lambda L) \\ \Rightarrow 1 - (1 - \lambda L)(1 + \lambda L) = \lambda^2 L^2. \end{array} . \tag{8}$$

- The next difficulty is to estimate the value of $K_2$.
- Our best guess for $K_2$ will be $\lambda^2 L^2$

$$\begin{gathered}(1-\lambda L)(1+\lambda L) = 1(1+\lambda L) - \lambda L(1+\lambda L) \\ \Rightarrow 1 + \lambda L - \lambda L - \lambda^2 L^2 = 1 - \lambda^2 L^2 \\ (1-\lambda L)\left(1+\lambda L+\lambda^2 L^2\right) = 1\left(1+\lambda L+\lambda^2 L^2\right) - \lambda L\left(1+\lambda L+\lambda^2 L^2\right) \\ \Rightarrow 1 + \lambda L + \lambda^2 L^2 - \lambda L - \lambda^2 L^2 - \lambda^3 L^3 = 1 - \lambda^3 L^3.\end{gathered} \tag{9}$$

- The division becomes

$$\begin{gathered}\frac{1}{1-\lambda L} = \left(1+\lambda L+\lambda^2 L^2 + K_3\right) \\ \Rightarrow 1 - (1-\lambda L)\left(1+\lambda L+\lambda^2 L^2\right) = \lambda^3 L^3\end{gathered} \tag{10}$$

  so by extension $K_3 = \lambda^3 L^3$.
- By induction we derive the result in Equation 6.

- Next, we want to consider a payment for a fixed period.
- First assume that the same payment will be made for four years starting at the beginning of each period.
- Assume a constant payment of $P$, the present value of this payment in the difference equation format becomes

$$PV = P\left(1 + \lambda L + \lambda^2 L^2 + \lambda^3 L^3\right). \tag{11}$$

Abstracting away from the payment, the formula can be reexpressed as

$$\begin{aligned}\left(1 + \lambda L + \lambda^2 L^2 + \lambda^3 L^3\right) &= \left(1 + \lambda L + \lambda^2 L^2 + \lambda^3 L^3 + \lambda^4 L^4 + \cdots\right) \\ &\quad -\lambda^4 L^4\left(1 + \lambda L + \lambda^2 L^2 + \lambda^3 L^3 + \cdots\right) \\ \left(1 + \lambda L + \lambda^2 L^2 + \lambda^3 L^3\right) &= \frac{1}{1 - \lambda L} - \lambda^4 L^4 \frac{1}{1 - \lambda L} = \frac{1 - \lambda^4 L^4}{1 - \lambda L}.\end{aligned} \tag{12}$$

- To make this into a present value formulation we let $\lambda = 1/\left(1+r\right) < 1$.
- Given that the payment is a constant $L \to 1$ so that Equation 12 implies

$$P\left(\frac{1-\lambda^4 L^4}{1-\lambda L}\right) = P\left(\frac{1-\left[\frac{1}{1+r}\right]^4}{1-\frac{1}{1+r}}\right). \qquad (13)$$

- Solving for the denominator of Equation 13

$$1-\frac{1}{1+r} = \frac{1+r}{1+r} - \frac{1}{1+r} = \frac{r}{1+r}. \qquad (14)$$

- Substituting Equation 13 into Equation 14 yields

$$
\begin{aligned}
P\left(\frac{1-\left[\frac{1}{1+r}\right]^4}{\frac{r}{1+r}}\right) &= P\left(\frac{1+r-\left[\frac{1}{1+r}\right]^3}{r}\right) \\
&= P\left(1+\frac{1-\left[\frac{1}{1+r}\right]^3}{r}\right)
\end{aligned}
\tag{15}
$$

- Generalizing Equation 15 for the general $n$ payments

$$PV = P\left(1 - \frac{1 - \left[\frac{1}{1+r}\right]^{n-1}}{r}\right) \tag{16}$$

which is the formula for an annuity due.

- The present value formula for an ordinary annuity is then

$$PV = P\left(\frac{1 - \left[\frac{1}{1+r}\right]^{n}}{r}\right). \tag{17}$$