Lecture XVI: Utility and the Time Value of Money

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1. The Mathematics of the Time Value of Money

The Mathematics of the Time Value of Money

- The **time value of money** refers to the concept that a dollar received today is worth more to a decision maker (or consumer) than a dollar received at some future time (consider one year ahead for convenience).
- Let's assume that utility is a function of two arguments, income received this year (Y_t) and income received next year (Y_{t+1}).
- Mathematically, we could write this function as $U(Y_t, Y_{t+1})$.
- We could then write the decision facing the consumer as

$$\begin{aligned} &\max_{Y_t, Y_{t+1}} U(Y_t, Y_{t+1}) \\ &\text{s.t. } F(Y_t, Y_{t+1}) \end{aligned} \tag{1}$$

where $F(Y_t, Y_{t+1})$ is the function which dictates how the individual must trade income between periods.

- If we assume a functioning capital market with a stated interest rate of r, Fisher's formulation gives the consumption choice as

$$\begin{gathered} \max_{Y_t, Y_{t+1}} U\left(Y_t, Y_{t+1}\right) \\ \text{s.t. } W \geq Y_t + \frac{1}{(1+r)} Y_{t+1} \end{gathered} \tag{2}$$

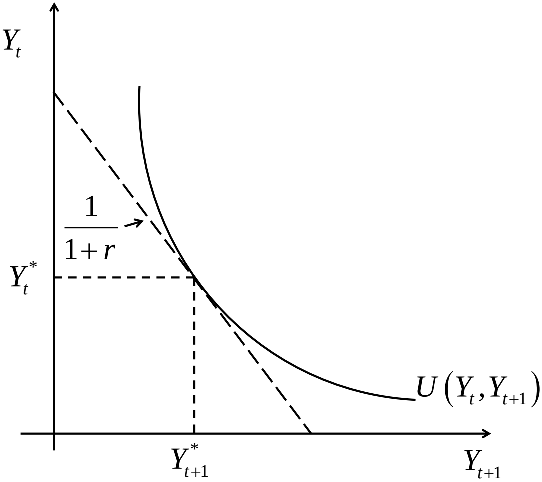
  where W is the wealth of the individual (i.e., the money in the bank).
- The individual faces the decision whether to consume this year or next year.

- Any consumption postponed to next year is increased by the interest rate paid (i.e., the price of future consumption is less than one $1/(1+r) < 1$ if $r > 0$). Using a little bit of calculus

$$\frac{\Delta Y_{t+1}}{\Delta Y_t} = 1 + r \tag{3}$$

or income at time $t+1$ has to increase by $1+r$ to be worth a dollar of income today.

Intertemporal Consumption Tradeoff



- While the graphics become cumbersome, it is relatively easy to extend the analysis mathematically to a decision between three periods.
- In Fisher's formulation the consumer's maximization problem becomes

$$U\left(Y_t, Y_{t+1}, Y_{t+2}\right)$$
$$\text{s.t. } W \geq Y_t + \frac{1}{1+r_1} Y_{t+1} + \frac{1}{1+r_2} Y_{t+2} \tag{4}$$

where r_1 and r_2 are two different interest rates.

- r_1 is associated with consumption at time $t+1$ and r_2 is the interest rate associated with consumption at time $t+2$.
- Thus, we assume that

$$\frac{1}{1+r_1} > \frac{1}{1+r_2}. \tag{5}$$

- The most common formulation is to assume a compound structure of interest such that r_2 contains the effect of the first period's interest rate

$$\frac{1}{1+r_2} = \frac{1}{1+r_1}\frac{1}{1+\tilde{r}_2} \tag{6}$$

where $\tilde{r}_2$ is the marginal interest rate between period $t+1$ and $t+2$.

- Substituting this expression back into Equation 4 yields

$$\begin{gathered} U\left(Y_t, Y_{t+1}, Y_{t+2}\right) \\ \text{s.t. } W \geq Y_t + \frac{1}{1+r_1}Y_{t+1} + \frac{1}{1+r_1}\frac{1}{1+\tilde{r}_2}Y_{t+2} \end{gathered} \tag{7}$$

Term Structure of Interest

- The relationship between r_1 and $\tilde{r}_2$ (and by extension between r_1 and $\tilde{r}_3$, $\tilde{r}_4$, ...) is referred to as the **term structure of the interest rate**.
- To investigate this question, let us start by assuming that $r_1 = \tilde{r}_2$ or that the interest rate between Y_{t+1} and Y_{t+2} is the same as the interest rate between Y_t and Y_{t+1}.
- Here price of consumption at time Y_{t+2} becomes

$$\frac{1}{1+r_1}\frac{1}{1+\tilde{r}_2} = \frac{1}{1+r_1}\frac{1}{1+r_1} = \frac{1}{(1+r_1)^2}. \tag{8}$$

- Another way to state this assumption is that the term structure of the interest rate is flat.

- An increasing term structure implies that the interest rate between Y_{t+1} and Y_{t+2} is expected to be higher than between Y_t and Y_{t+1}, or

$$\frac{1}{1+r_1}\frac{1}{1+\tilde{r}_2} < \frac{1}{(1+r_1)^2}. \tag{9}$$

- This increasing term structure is somewhat normal if investors believe that risk is increasing over time.

Weighted Average Cost of Capital

- The most widely accepted method of choosing a discount rate is to set it equal to the firm's **weighted average cost of capital**.
- Table 1 presents the data for computing the weighted average cost of capital for a firm with the financial characteristics of the Indiana farm depicted in Chapter 3.

Weighted Average Cost of Capital for Indiana Farm

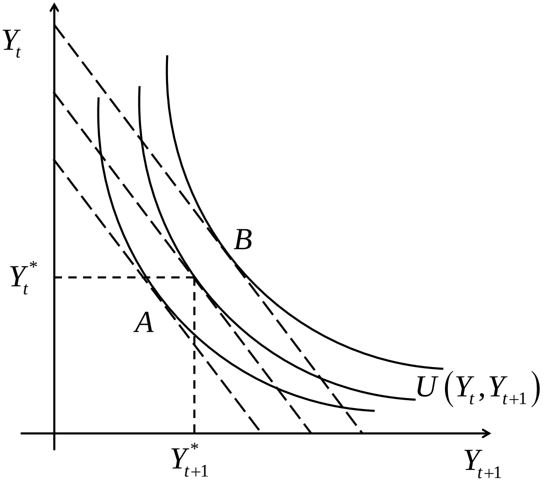
Equity Source	Interest Rate	Total Amount	Share of Equity	Result
Non Real Estate Debt				
Operating Loan	15	51,900	0.0208	0.3114
Equipment Loan	10	77,850	0.0311	0.3114
Real Estate Loan				
Farm Credit System	12	550,000	0.2200	2.6400
Owner's Equity	4	1,820,250	0.7281	2.91924
Total Assets		2,500,000		
Total Interest Rate			6.1752	

- Consider whether to purchase an investment that will cost \$1,000 this year and pay \$1,125 next year.
- To integrate this investment into our two period formulation in Equation 2, we let $\tilde{Y}_t = -1,000$ and $\tilde{Y}_{t+1} = 1,125$.
- The result of this investment will be some change in wealth $\tilde{W}$.

$$\begin{gathered} \max_{Y_t, Y_{t+1}} U\left(Y_t, Y_{t+1}\right) \\ \text{s.t. } W + \tilde{W} \geq \left(Y_t + \tilde{Y}_t\right) + \frac{1}{(1+r)}\left(Y_{t+1} + \tilde{Y}_{t+1}\right) \end{gathered} \tag{10}$$

- The decision maker will choose the investment if the utility resulting from Equation 10 is greater than the utility from Equation 2.

Making an Investment



- One characteristic of Figure 2 implied by Equation 10 is that the budget constraints are parallel.
- The question of whether the producer is better off or not can be answered by simply computing the change to the decision maker's wealth constraint

$$\tilde{W} = \tilde{Y}_t + \frac{1}{(1+r)}\tilde{Y}_{t+1}. \tag{11}$$

- If $\tilde{W} > 0$, the investment makes the decision maker better off.
- If $\tilde{W} < 0$, the decision maker would be made worse off making the investment.
- For the current scenario

$$\tilde{W} = -1,000 + \frac{1,125}{(1+0.08)} = 41.67 \tag{12}$$

- We can extend the usage of the preceding example to three periods by simply considering the effect of an investment on wealth over three periods

$$\tilde{W} = \tilde{Y}_t + \frac{1}{(1+r)}\tilde{Y}_{t+1} + \frac{1}{(1+r)^2}\tilde{Y}_{t+2} \tag{13}$$

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